TREE GROUPS OF SHIFTED GRAPHS

PAUL BENDICH AND TRISTRAM BOGART

Abstract. The tree group is a subtle algebraic invariant of a graph, derivable from its Laplacian matrix. We describe these objects in general and then detail our efforts at classifying the group for the special case of shifted graphs.

1. Laplacian Matrices of Graphs

Let G be graph on n vertices with m edges, with no loops or multiple edges allowed. Label its vertices $v_1, \dots, v_n$. Its *Laplacian* $L(G) = (\ell_{ij})$ is the $n \times n$ matrix given by

$$l_{ij} = \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{if } i \neq j \text{ and } v_i \text{ is not adjacent to } v_j \end{cases}$$

For example, let G_1 be the simple graph on four vertices consisting of a triangle on v_1, v_2, and v_3, with another vertex v_4 connected only to v_1. Then $L(G_1)$ is the 4×4 matrix below.

$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

For more on Laplacians, see [**?**].

2. Tree Groups

For any graph G, $L(G)$ is an integer matrix, so it can be interpreted as a linear transformation $L : \mathbf{Z}_n \to \mathbf{Z}_n$. Then Im L is a subgroup of $\mathbf{Z}_n$, and so is its quotient group $(\mathbf{Z}_n)/(\text{Im } L)$. This quotient group is called the *tree group* of G (also called the *Picard group* or the *Jacobian* in [**?**]). Unfortunately, L is a singular matrix, as each row and each column sum to 0, so its image is not of full rank, and the tree group includes at least one factor of $\mathbf{Z}$. The number of such factors is the number of connected components in G. It is possible to get around this by defining a nonsingular *reduced Laplacian*, formed by striking out one row and column in the section formed from each component of G, but we will simply ignore the unwanted factors when we want to think of the tree group as finite, e.g. in the *Matrix-Tree Theorem*:

Theorem 1. [**?**] *The cardinality of the tree group (ignoring any factors of* $\mathbf{Z}$*) of a connected graph* G *is the number of spanning trees in* G*.*

This work was done as part of an REU during the summer of 2000 at the University of Minnesota—Twin Cities under the direction of Vic Reiner.

Corollary 2. *The cardinality of the tree group of a disconnected graph G is the product of the numbers of spanning trees in the various components of G, or equivalently the number of spanning forests in G.*

The corollary holds because if A and B are two components of G, then the rows/columns of L corresponding to vertices in A contain only 0's in columns/rows corresponding to vertices in B, so the section of L associated with A can be reduced to a *Smith Normal Form* (see below) without affecting the section associated with B. For example, if G consists of two disjoint triangles, then $L(G)$ is as follows:

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

The Laplacian is always symmetric, so the row space is the same as the column space. Furthermore, different vertex orderings only permute the rows and columns of the Laplacian, which corresponds to permuting the bases with respect to which the Laplacian represents the transformation. Therefore the image of the transformation, and thus the tree group, are invariants of G (up to isomorphism).

To calculate the tree group of a graph, we create the *Smith Normal Form* $S(G)$ of the Laplacian. The Smith Normal Form of an $n \times n$ integer matrix is a diagonal matrix with each diagonal entry dividing the next. It is created by multiplying the matrix on the left and right by *unimodular* $n \times n$ matrices. A unimodular integer matrix is one that is invertible within the ring of integers; equivalently, one of determinant ± 1. So for any matrix A,

$$S(A) = V_1 A V_2$$

where V_1 and V_2 are unimodular. Right multiplication by unimodular matrices represents compositions of a limited set of row operations: adding an integer multiple of a row to another row or permuting rows, and left multiplication represents the same operations on columns. Every matrix has a unique Smith Normal Form, a fact most often proved and used in the classification theorem of finitely generated abelian groups (see e.g. [**?**, Chap. 12]).

Proposition 3. *For any $n \times n$ matrix A with Smith Normal Form S, Im $A \cong$ Im S.*

Proof. Since V_1 and V_2 are unimodular, we can multiply the last equation through by their inverses to obtain

$$A = U_1 S U_2$$

where U_1 and U_2 are also unimodular. Then

$$\begin{aligned} x \in \text{Im } A &\iff \exists y \text{ s.t. } Ay = x \\ &\iff U_1 S U_2 y = x \\ &\iff S U_2 y = U_1^{-1} x \end{aligned}$$

Since U_2 is invertible, it represents a bijection from $\mathbf{Z}_n$ to $\mathbf{Z}_n$. So as y ranges through $\mathbf{Z}_n$, so does $U_2 y$. Similarly, U_1^{-1} also represents a bijection, so as x ranges through $\mathbf{Z}_n$, so does $U_1^{-1}x$. Thus $x \in$ Im A if and only if $U_1^{-1}x \in$ Im S. So U_1^{-1} gives an isomorphism Im $A \to$ Im S, with inverse U_1. □

Once we have the Smith Normal Form, then, the tree group can easily be read off the diagonal entries; an integer vector $[x_1, x_2, \ldots, x_p]$ is in the image of a diagonal matrix with diagonal entries $[d_1, d_2, \ldots, x_p]$ if and only if for all i, x_i is a multiple of d_i. If a diagonal entry d_i is 0, x_i must always be 0. Assuming there are r 0's along the diagonal, the tree group, quotient group of this image, is:

$$\mathbf{Z}_{d_1} \times \mathbf{Z}_{d_2} \times \cdots \times \mathbf{Z}_{d_{n-r}} \times \mathbf{Z}^r$$

.

So for any given graph of reasonably small size, the tree group can be easily computed, e.g. in Maple using the command `ismith`. However, it is known in general for very few classes of graphs. The tree group of any tree is the trivial group since a tree obviously has exactly one spanning tree.

Theorem 4. [?] *The tree group of a complete graph K_n is $(\mathbf{Z}_n)^{n-2}$.*

Theorem 5. [?] *The tree group of a complete bipartite graph $K_{p,q}$ is $(\mathbf{Z}_p)^{p-2} \times (\mathbf{Z}_q)^{q-2} \times \mathbf{Z}_{pq}$.*

Theorem 6. [?] *The tree group of a cycle C_n is $\mathbf{Z}_n$.*

All of these can be proved by matrix manipulation, which is fairly straightforward at least in the first two cases.

The orders of the individual cyclic groups whose product is the tree group are called the *invariant factors* of the group. As noted, these are also the diagonal entries of the Smith Normal Form, and so the 1's that appear along this diagonal are also often considered invariant factors; including them in the product is harmless.

3. Shifted graphs

For our research, we focused on the class of *shifted graphs*. A simple graph G is *shifted* (or, in some sources, *threshold* or *degree-maximal*) if for any two vertices v, w in G with $\deg(v) \geq \deg(w)$, every neighbor of w is either v or a neighbor of v. There are several other equivalent definitions. For instance, shifted graphs are the class of all simple graphs that can be generated by beginning with a single vertex and then adding any number of vertices one at a time, with the restriction that each vertex when it is added must be given either no edges at all or an edge to every other vertex already in the graph.

Another way we found to define connected shifted graphs (or components of disconnected ones) is that they are exactly those graphs that can be broken up into a collection of disjoint sets of vertices $K_1, K_2, \ldots, K_r, I_1, I_2, \ldots, I_s$ such that:

- K_i is a clique for $i = 1, 2, \ldots, r$.
- I_j is an independent set for $j = 1, 2, \ldots, s$.
- Either $r = s$ or $r = s + 1$.
- Any two vertices in two different cliques are adjacent.
- Any two vertices in two different independent sets are not adjacent.
- If $v_i \in K_i$ and $v_j \in I_j$, then v_i and v_j are adjacent iff $i \leq s - j$.

We note that all vertices in a set have the same degree, and the given ordering of sets represents a strictly decreasing order of degrees. In our example graph, we have $r = 2$, $s = 1$, $K_1 = \{v_1\}$, $K_2 = \{v_2, v_3\}$, and $I_1 = \{v_4\}$.

Proposition 7. *A shifted graph is uniquely determined by its degree sequence.*

This can easily be seen from our classification or somewhat less easily from the first given definition.

4. Ferrers Diagrams and Merris' Theorem

A *Ferrers diagram* for a graph G consists of n rows of left-justified squares (rendered here by $\times$ signs), each row representing a vertex. All the rows are left-aligned, and the number of squares in row i is $\deg(v_i)$. Here's the diagram for our main example:

$$\begin{array}{l} \times\times\times \\ \times\times \\ \times\times \\ \times \end{array}$$

Russell Merris [**?**] calculated the eigenvalues of $L(G)$ for a shifted graph, which implies the following key fact about shifted graphs:

Theorem 8. *If G is a shifted graph, then the order of the tree group of G is the product of all column lengths except the first in the Ferrers diagram of G.*

5. Our Explorations

Merris's theorem was the starting point for our work. Noting that complete graphs are shifted, and that the invariant factors, apart from an extra 1, are the same as the column lengths (other than the first one, which by Merris's Theorem is to be ignored), we initially conjectured that this situation holds for all shifted graphs. However, we quickly found counterexamples; one small one has degree sequence $d = (3, 3, 2, 2)$. The column lengths in its Ferrers diagram are 4, 4, and 2 and its tree group is not $\mathbf{Z}_4 \times \mathbf{Z}_2$ but $\mathbf{Z}_8$. We then began using our new definition of a shifted graph, further defining a *k-block* (shifted) graph as one in which the total number of cliques plus independent sets, $r + s$, is k. A 1-block shifted graph is complete, and vice versa.

We then considered the 2-block case, whose graphs consist of one clique K and one independent set I and include every possible edge between the two sets. The counterexample to the original conjecture belongs to this case. Letting k and i be the numbers of vertices in K and I, respectively, the degree sequence d of a 2-block graph is

$$(\underbrace{n-1,\ldots,n-1}_{k},\underbrace{k,\ldots,k}_{i}).$$

(Recall that n is the total number of vertices; here, it is $k + i$.) The Ferrers column length sequence F consists of $k - 1$ n's and i k's. By reducing the appropriate general Laplacian by hand, we proved that:

Theorem 9. *The tree group of a 2-block shifted graph is $(\mathbf{Z}_n)^{k-2} \times (\mathbf{Z}_k)^{i-2} \times \mathbf{Z}_{nk}$.*

This form of the group is not necessarily the canonical one that would be read directly from the Smith Normal Form. For example, if the graph consists of a clique of three vertices and an independent set of four vertices, then the column lengths consist of three 7's and three 3's. According to our theorem, the tree group is $\mathbf{Z}_7 \times \mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_{21}$. This is true, but 3, 3, 7, and 21 cannot be ordered in any way such that each one in the list divides the next; the canonical form of this group is $\mathbf{Z}_3 \times \mathbf{Z}_{21} \times \mathbf{Z}_{21}$. In general, this means that our conjectures only state what the group factors always *can* consist of or contain, not what they *must*.

The number of factors should always be $n-1$, the dimension of the reduced Laplacian, so two 1's should be included among them in the 2-block case. In future cases, the necessary 1's will be included in the notation of the group.

After proving our 2-block theorem, we proceeded to 3-block graphs, which consist of two cliques and one independent set, but here two different cases emerged, depending on whether or not the independent set contains at least two vertices. The proofs were again by matrix manipulation, and relied on some seemingly arbitrary steps discovered only by much trial and error:

Theorem 10. *Let G be a 3-block shifted graph consisting of two cliques K_1 and K_2 and an independent set I, of sizes k_1, k_2, and i, respectively. Then the tree group is isomorphic to*

$$\begin{cases} (\mathbf{Z}_n)^{k_1-2} \times (\mathbf{Z}_{k_1+k_2})^{k_2-2} \times (\mathbf{Z}_{k_1})^{i-2} \times \mathbf{Z}_{nk_1} \times \mathbf{Z}_{(k_1+k_2)k_1} \times (\mathbf{Z}_1)^3 & \textit{if } i>1 \\ (\mathbf{Z}_n)^{k_1-2} \times (\mathbf{Z}_{k_1+k_2})^{k_2-2} \times \mathbf{Z}_{nk_1(k_1+k_2)} \times (\mathbf{Z}_1)^3 & \textit{if } i=1 \end{cases}$$

We note that the second case doesn't contradict the spirit of the first, since when $i=1$, the requirement in the first case of $i-2$ factors of $\mathbf{Z}_{k_1}$ doesn't make sense. This raises the question of what happens when k_1 or k_2 is 1. However, it is not possible for k_2 to be 1 because one vertex in a "clique" by itself, connected to the other clique but to nothing else, is indistinguishable from the vertices of the independent set. We would then say that instead of two cliques and an independent set, the graph has one clique of size k_1 and one independent set of size $i+1$, and classify it in the 2-block case.

If k_1 is *1*, then vertices in the last independent set are connected only to the single vertex in K_1; thus they are leaves. Adding a leaf to a graph doesn't really change the number or the nature of its spanning trees; the new edge at the leaf must simply be added to each tree. Furthermore, removing leaves will leave a shifted graph shifted. So we were not surprised to find that:

Proposition 11. *The tree group of a shifted graph remains unchanged if leaves are removed.*

Starting from the Laplacian of such a graph, it is easy to create 1's on the diagonal in the positions corresponding to leaves and the vertex connected to the leaves and 0's elsewhere in their rows and columns without affecting the rest of the matrix. So a graph with $k_1 = 1$ has the same tree group as one without the leaves that are connected to k_1, so that there is one fewer clique, one fewer independent set, and one more vertex in the first listed clique remaining, K_2.

Before we could conjecture anything about the general case, we had to realize that the column lengths in the Ferrers diagram are the same as the vertex degrees for vertices in independent sets, and one greater for vertices in cliques. So we extend our notation and allow K_i to represent the set of *columns* of length k_i+1, and similarly I_j to represent the set of columns of length i_j. The total number of columns, however, is one less than the total number of rows; the discrepancy appears in K_r in odd-block cases and in I_1 in even-block cases. Apart from this, the number of columns of a particular length x always matches the number of vertices of the corresponding degree (which, to recap, is x if the vertices are in an independent set or $x-1$ if they are in a clique).

Now we can consider our results about the tree group in terms of combinations of column lengths. In the complete graph, there were no combinations at all; the invariant factors are all identical to column lengths. In the 2-block case, there was one combination of a column in K with one in I. In the 3-block case with at least two vertices in the independent set, one column in K_1 combined with a column in I, while another column in I combined with one in K_2. Implicit in such thinking is the conjecture that:

Conjecture 12. *The tree group can be written in the form $\prod_i \mathbf{Z}_{d_i}$ where the d_i's are 1's, column lengths, and products of column lengths. Column lengths need not be factored into two numbers that contribute to two different d_i's.*

For example, with 2 columns, both of length 4 (assume the first column has already been removed), possible tree groups would be $\mathbf{Z}_4 \times \mathbf{Z}_4$ and $\mathbf{Z}_{16}$, but not $\mathbf{Z}_2 \times \mathbf{Z}_8$.

In testing other conjectures, we implicitly tested this one with several dozen graphs and found no counterexamples.

After proving our 3-block theorems, we conjectured that if there are sufficient numbers of columns of each length, then one column representing each clique combines with one representing each independent set. We proved this false with 4-block examples, in which we found that while the conjecture otherwise held, its predicted combination of a column from I_1 with one from K_1 did not occur. After trying some even larger graphs, we now believe:

Conjecture 13. *Arrange the column sets in the following path:*

$$K_1 \leftrightarrow I_s \leftrightarrow K_2 \leftrightarrow I_{s-1} \leftrightarrow \ldots \leftrightarrow I_2 \leftrightarrow K_r \leftrightarrow I_1$$

for an even-block graph or

$$K_1 \leftrightarrow I_s \leftrightarrow K_2 \leftrightarrow I_{s-1} \leftrightarrow \ldots \leftrightarrow K_{r-1} \leftrightarrow I_s \leftrightarrow K_r$$

for an odd-block one. Also assume that the each blocks with two neighbors in the path corresponds to a set of at least 2 columns.

Then one combination occurs between columns from each pair of adjacent sets in the appropriate diagram above.

This conjecture has some nice corollaries; the number of 1's among the group factors in a k-block graph is always k, and for any clique/independent set consisting of $p > 1$ vertices of degree q, the number of $q+1$'s/q's among the factors is exactly $p-2$. The first of these is nearly proved by our following observation:

Proposition 14. *In a k-block graph with at least two vertices in each block, there are at least k 1's in the Smith Normal Form of its Laplacian.*

The reason that this does not quite prove what we want about the 1's is that we want to know how many 1's occur in the expression of the group predicted by our conjecture, which we have noted is not necessarily the same as the expression read directly off the Smith Normal Form. But the number of 1's in the Smith form of a group is an upper bound for the number of them in any form, so this proposition makes our conjecture far more plausible.

Proof. We will find a $k \times k$ minor of the Laplacian with determinant ± 1. Choose vertices $v_1, \ldots, v_n$, one from each of the n blocks and also $w_1, \ldots, w_n$ in the same way, with $v_i \neq w - i$ for all i. Then consider the minor with rows $v_1, \ldots, v_n$ and columns $w_1, \ldots, w_n$. Below and to the right of the secondary (upper-right to lower-left) diagonal every entry is 0, so the determinant is simply -1 times the product of the entries along this diagonal, all of which are -1.

Once we have this minor, we can reduce it to a diagonal matrix of 1's, and then use these to clear everything else in their rows and columns in the full Laplacian. These 1's will remain as we reduce the rest of the matrix. □

An extension of this proposition deals with some cases where there is only one vertex in a block.

Corollary 15. *The conclusion of the proposition only requires that the independent sets, not the cliques, contain at least two vertices each.*

Proof. If a clique K_i contains only one vertex, we must choose $v_i = w_i$. But this only affects the (i,i)th entry in the matrix, which is along the main diagonal and, because the number of cliques is equal to or one number than the number of independent sets, is not in the lower right section of the minor, which we need to consist entirely of 0's. Altering entries in the upper left section will not affect the determinant. The only possible problem is if an altered entry is along the secondary diagonal; however, the two diagonals only intersect if n is odd, and then they do so in position (r,r); recall that K_r is the last clique. As we observed earlier, in an odd-block case this clique cannot consist of a single vertex; otherwise it would be indistinguishable from the first independent set. □

Again, this shows that the tree group for a k-block graph, this time with some cliques consisting of a single vertex, can be expressed in a form including k 1's.

References

[1] Artin, Michael. *Algebra.* Prentice-Hall, Inc., Englewood Cliffs, NJ, 1991.

[2] Biggs, Norman. Algebraic potential theory on graphs, *Bulletin of the London Mathematical Society* **29** (1994), no. 6, 641-682.

[3] Lorenzini, Dino. A finite group attached to the laplacian of a graph, *Discrete Mathematics* **91** (1991), 277-282.

[4] Merris, Russell. Degree-maximal graphs are Laplacian integral, in *Linear Algebra and its Applications* **199** (1994), 381-389.

[5] Merris, Russell. Laplacian matrices of graphs: a survey, in *Linear Algebra and its Applications* **197/198** (1994), 143–176.

[6] Merris, Russell. Unimodular equivalence of graphs, in *Linear Algebra and its Applications* 173 (1992), 181-189.

E-mail address: bendich@grinnell.edu

E-mail address: tristram.bogart@oberlin.edu